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Stability and dynamics of a harmonically excited elastic-perfectly plastic oscillator

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Abstract

This paper deals with the stability and the dynamics of a harmonically excited elastic-perfectly plastic oscillator. The hysteretical system is written as a non-smooth forced autonomous system. It is shown that the dimension of the phase space can be reduced using adapted variables. Free vibrations of such a system are then considered for the damped system. The extended direct method of Liapounov is applied to this non-smooth mechanical system and the asymptotic stability of the origin is proven in the new phase space. The forced vibration of such an oscillator is treated by numerical approach, by using the time locating techniques. The stability of the limit cycles is analytically investigated with a perturbation approach. The boundary between elastoplastic shakedown and alternating plasticity is given in closed form. It is shown that this boundary corresponds to a bifurcation boundary for the undamped system (period-doubling bifurcation). Finally, the equivalent damping of this hysteretic system is characterized from dynamical properties.

1. Introduction

The problem of designing reliable civil engineering structures to resist to strong earthquake ground motions leads naturally to the investigation of the dynamic behaviour of structures beyond the yield limit. The elastoplastic behaviour is probably one of the most commonly used constitutive laws of inelasticity in the nonlinear range. For fundamental analysis and understanding of the dynamics of such inelastic structures, a single-degree-of-freedom system is often considered. As a consequence, dynamics of an elastoplastic single-degree-of-freedom system may constitute a generic structural case that is very useful in the analysis of more complex structures. This paper is devoted to the stability and the dynamics of a harmonically excited elastic-perfectly plastic oscillator. It can be recalled that perfect plasticity only means that the yield force (or maximum force in the elastic domain) remains constant during the plastic flow. Of course, perfect plasticity is a particular case of hardening plasticity (in this case, the yield force may evolve during the plastic flow) and consequently the literature dealing with hardening plastic oscillators also includes perfect plastic oscillators (see Ref. [1] for the synthesis of main constitutive laws). It could be quite surprising that a specific paper would be focused nowadays on an apparently so simple system. The motivations are first to analyse the dynamics of

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such an oscillator from a new point of view, using a very common technique in the field of non-smooth dynamics [2]. Secondly, the paper presents some fundamental phenomena associated to the theory of shakedown of elastoplastic structures. Shakedown can be defined as the capability for the oscillator to converge towards a stationary elastic regime (stationary regime without plastic phases). This property is very important for the structural system to be guaranteed, in order to achieve reliable engineering design (the reader should refer to [3] for an extensive description of this problem in quasi-static analysis). Finally, the connection between equivalent damping and dynamical properties of the elastoplastic system will be discussed.

There have been numerous studies on the forced response of elastoplastic oscillators using a bilinear hysteretic model (also called linear kinematic hardening rule). Jacobsen [4] and Tanabashi [5] are among the earlier investigators studying the forced vibration response of yielding oscillators to simple pulses and square waves. Caughey [6] was the first to obtain an approximation of the steady-state response of the undamped bilinear hysteretic oscillator subjected to harmonic pulsation. Caughey [6] uses the method of slowly varying parameters to approximate the response and he investigates the stability of the postulated periodic evolutions. One of the main results of Caughey's reference study is that the steady-state response of this oscillator was stable for all the involved parameters. Jennings [7] or Iwan [8] generalized the results of Caughey [6] by considering more complex hysteretic models and by adding some damping. The first considers Ramberg–Osgood's model whereas the second studies a particular hysteretic model called the double bilinear hysteretic model. More recently, Capecchi [9] studied the Bouc's hysteresis model. It is not the purpose of this paper to enumerate all hysteresis models and their dynamics properties.

The important point is that most of these studies use mainly approximate analytical techniques such as averaging method [6–8], harmonic balance method [10] or by combining Fourier transform and harmonic balance technique [9]. The latter is probably one of the most efficient. However, the accuracy of the results depends, like all harmonic balance techniques, on the number of harmonics included. These problems become more prominent if the response contains sharp spikes (i.e. due to the sharp change in the restoring force at the elastic–plastic boundary) when a large number of harmonics need to be retained (Chatterjee et al. [11]). In the present paper, it is shown that the dynamics of the elastoplastic oscillator is a forced piecewise linear system. The dynamics is computed using the time locating techniques, as initiated by Masri [12] for impact dampers. The stability analysis is carried out by using a perturbation approach, first applied to impact dampers (Masri and Caughey [13]). Due to the non-smooth nonlinearities, the stability of the periodic solution is determined by investigating the asymptotic behaviour of perturbations to the steady state periodic solution, as the usual method involving the classical Floquet theory is not applicable to such a non-smooth system. This method also called the method of error propagation [11] needs the accurate knowledge of the number of junctions to be encountered during one period. This method was also applied by Miller and Butler [14] or Capecchi [15] to the undamped elastoplastic oscillator. The damped elastoplastic oscillator was studied by Masri [16] who investigates the exact steady-state motion by using the piece-wise linear properties of the non-smooth dynamical system. Masri [16] extended the pioneering work of Iwan [17] who also determined the exact steady-state motion of the undamped system. Coman [18,19] questioned the boundedness of trajectories of the damped elastoplastic oscillator. Dynamics of the damped forced elastoplastic oscillator has been recently considered by Liu and Huang [20] who have obtained a closed-form solution of the exact steady-state motion [20]. Moreover, they have shown for a specific range of parameters that this steady-state motion is not unique. We will discuss this result later in the paper. It is interesting to notice that dynamics of elasto-plastic oscillators, dynamics of friction oscillators or dynamics of impact dampers are finally closely connected by similar analysis techniques, due to their non-smooth nature. Recent contributions to the stability analysis of these periodically forced piecewise linear oscillators are those of Wiercigroch [21], Awrejcewicz and Lamarque [2], Ji and Leung [22] or Luo and Chen [23].

In this paper, the presented perturbation method is applied to the harmonically excited elastic-perfectly plastic oscillator. In the first part, the dynamic hysteretic system is written as a non-smooth forced autonomous system. It is shown that the dimension of the phase space can be reduced using adapted variables. Free vibrations of such a system are then considered and the asymptotic stability of the origin in the new phase space is shown. However, this fixed point depends on the initial conditions in the initial phase space.

The forced vibration of such an oscillator is treated by numerical approach, by using the time locating technique. The stability of the limit cycles obtained for a certain range of structural parameters (damping, force intensity and excitation frequency) is then analytically investigated. A bifurcation boundary is highlighted in the space of structural parameter and connected to the shakedown-alternating plasticity boundary (Challamel [24]).

2. Equations of motion

Let us consider the simple system shown in Fig. 1. A mass M is attached to a viscous elasto-plastic spring. The inelastic system is externally excited by a harmonic force $F(t)$ defined by the intensity F_0, and angular frequency Ω. t is the time and a superposed dot represents a time differentiation. This oscillator is characterized by the displacement U, displacement rate $\dot{U}$, and a plastic internal variable, chosen as the plastic displacement U_p. The plastic incremental law is illustrated in Fig. 2. This is a simple elastoplastic perfect law, which depends on two parameters, i.e., the elastic stiffness K_0 and the maximum force F^+. U_Y is the elastic displacement, limit of the initial virgin state ($U_Y = F^+/K_0$). The damping coefficient is denoted by C.

Two dynamic states can be distinguished. These two states correspond to a reversible state $\hat{E}$ (or elastic state) and an irreversible state $\hat{P}$ (or plastic state), associated with plastic displacement evolution. This plastic state $\hat{P}$ can be decomposed into two states $\hat{P}^+$ and $\hat{P}^-$, depending on the sign of the elastic displacement $U - U_p$. The equation of motion of the damped elasto-plastic oscillator can be written as

$$\begin{aligned}
&\hat{E} \text{ state}: \; M\ddot{U} + C\dot{U} + K_0(U - U_p) = F(t); \; \dot{U}_p = 0, \\
&\hat{P}^+ \text{ state}: \; M\ddot{U} + C\dot{U} + F^+ = F(t); \; \dot{U}_p = \dot{U}, \\
&\hat{P}^- \text{ state}: \; M\ddot{U} + C\dot{U} - F^+ = F(t); \; \dot{U}_p = \dot{U} \quad \text{with } F(t) = F_0 \cos \Omega t. \qquad (1)
\end{aligned}$$

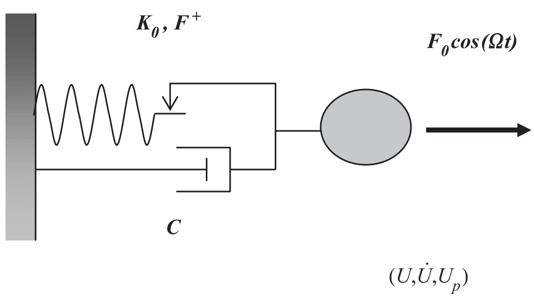

Fig. 1. Description of the elastic-perfectly plastic oscillator.

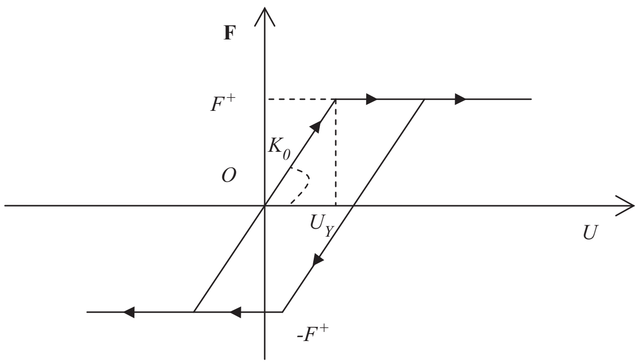

Fig. 2. Plastic incremental law for the inelastic spring.

Each state is defined from a partition of the phase space $(U, \dot{U}, U_p)$:

$$\begin{aligned}
&\hat{E}\text{ state}: \ (|U-U_p|<U_Y) \quad \text{or} \quad \left[(|U-U_p|=U_Y) \quad \text{and} \quad (\dot{U}(U-U_p)\leqslant 0)\right],\\
&\hat{P}^+\text{ state}: \ (U-U_p=U_Y) \quad \text{and} \quad \dot{U}\geqslant 0,\\
&\hat{P}^-\text{ state}: \ (U-U_p=-U_Y) \quad \text{and} \quad \dot{U}\leqslant 0.
\end{aligned} \tag{2}$$

This is clearly a piecewise linear oscillator and a non-smooth system (as in Shaw and Holmes [25]). The dimensionless phase variables are introduced as follows:

$$\left(u, \dot{u}, u_p\right) = \left(\frac{U}{U_Y}, \frac{\dot{U}}{U_Y}, \frac{U_p}{U_Y}\right). \tag{3}$$

The time constant of the dynamical system is introduced as

$$t^* = \sqrt{\frac{M}{K_0}}. \tag{4}$$

New temporal derivatives can be calculated with respect to the dimensionless time parameter

$$\tau = \frac{t}{t^*}. \tag{5}$$

It means that the dot now represents a time differentiation with respect to dimensionless variable τ. The equations of motion can now be formulated using the dimensionless variables:

$$\begin{aligned}
&\hat{E}\text{ state}: \ \ddot{u}+2\varsigma\dot{u}+(u-u_p)=f_0\cos\omega\tau, \quad \dot{u}_p=0,\\
&\hat{P}^+\text{ state}: \ \ddot{u}+2\varsigma\dot{u}+1=f_0\cos\omega\tau, \quad \dot{u}_p=\dot{u},\\
&\hat{P}^-\text{ state}: \ \ddot{u}+2\varsigma\dot{u}-1=f_0\cos\omega\tau, \quad \dot{u}_p=\dot{u}\\
&\quad \text{with } f_0=F_0/F^+, \quad \omega=\Omega\sqrt{M/K_0}, \quad \varsigma=C/2\sqrt{MK_0}.
\end{aligned} \tag{6}$$

ς is the damping ratio (positive parameter). The three states are now governed by

$$\begin{aligned}
&\hat{E}\text{ state}: \ (|u-u_p|<1) \ \text{or} \ \left[(|u-u_p|=1) \quad \text{and} \quad (\dot{u}(u-u_p)\leqslant 0)\right],\\
&\hat{P}^+\text{ state}: \ (u-u_p=1) \quad \text{and} \quad \dot{u}\geqslant 0,\\
&\hat{P}^-\text{ state}: \ (u-u_p=-1) \quad \text{and} \quad \dot{u}\leqslant 0.
\end{aligned} \tag{7}$$

The dimension of the phase space can be reduced using the elastic displacement variable

$$v = u - u_p. \tag{8}$$

The new phase space is then reduced to $(v, \dot{u})$. This is an important property of this perfectly plastic oscillator. Such a reduction of the dimension of the phase space would not be possible for the kinematic hardening plastic oscillator for instance. This property significantly simplifies the calculations for the investigations of the steady-state motion and for the stability analysis. It can be remarked that the elastic displacement is nothing else than the constitutive force for the dimensionless dynamical system. This reduced phase space was also used by Capecchi [15] or Coman [18]. Thus, the new equivalent dynamical system is now given by

$$\begin{aligned}
&\hat{E}\text{ state}: \ \ddot{u}+2\varsigma\dot{u}+v=f_0\cos\omega\tau, \quad \dot{v}=\dot{u},\\
&\hat{P}^+\text{ state}: \ \ddot{u}+2\varsigma\dot{u}+1=f_0\cos\omega\tau, \quad \dot{v}=0,\\
&\hat{P}^-\text{ state}: \ \ddot{u}+2\varsigma\dot{u}-1=f_0\cos\omega\tau, \quad \dot{v}=0.
\end{aligned} \tag{9}$$

Correspondingly, the three states in the $(v, \dot{u})$ phase space are

$$
\begin{aligned}
&\hat{E}\ \text{state}:\ (|v|<1) \quad \text{or} \quad [(|v|=1) \quad \text{and} \quad (\dot{u}v \leqslant 0)],\\
&\hat{P}^{+}\ \text{state}:\ (v=1) \quad \text{and} \quad \dot{u} \geqslant 0,\\
&\hat{P}^{-}\ \text{state}:\ (v=-1) \quad \text{and} \quad \dot{u} \leqslant 0.
\end{aligned} \tag{10}
$$

These two systems Eqs. (9) and (10) can be merged into one non-smooth system by using non-smooth functions (as suggested by Chicurel-Uziel [26] for piecewise linear springs):

$$
\begin{aligned}
&\ddot{u} = -2\varsigma\dot{u} - v + f_0 \cos \omega\tau,\\
&\dot{v} = |\dot{u}|[h(1-v) - h(1+v)] + \dot{u}h(1-|v|)
\end{aligned}
\quad \text{with} \ \ h(x) = \begin{cases} 1 & \text{if} \ \ x>0, \\ \frac{1}{2} & \text{if} \ \ x=0, \\ 0 & \text{if} \ \ x<0. \end{cases} \tag{11}
$$

Here, h is the Heaviside step function. The non-smooth character of such a system is no longer ambiguous with this unified presentation.

3. Free vibrations

For $f_0 = 0$, i.e. for free vibration, the dynamic system is an autonomous system with a two-dimensional phase space associated with the coordinates $(v, \dot{u})$. This conversion from the inelastic system (and hysteretical system) to an autonomous system was soon noticed by adding internal variables (see Savi and Pacheco [27] or more recently Challamel and Pijaudier-Cabot [28] for damage systems or Challamel and Pijaudier-Cabot [29] for plastic softening systems). The asymptotic stability of the origin point will be shown with two different methods, first by solving the evolution problem, and secondly by using the extended direct method of Liapounov. We have to mention that the application of the extended direct method of Liapounov (for non-smooth systems) to elastoplastic systems, has not been reported in the literature, to the authors knowledge.

3.1. Evolution problem

The initial time is arbitrarily chosen to vanish ($\tau_0 = 0$). It may be assumed that the perturbation first leads to an elastic regime such as

$$
\begin{aligned}
&\ddot{v} + 2\varsigma\dot{v} + v = 0,\\
&\dot{u} = \dot{v} \quad \text{with} \quad v(\tau_0 = 0) = v_0, \ \ \dot{u}(\tau_0 = 0) = \dot{u}_0 = \dot{v}_0.
\end{aligned} \tag{12}
$$

The dynamic response is simply obtained by

$$
\begin{aligned}
v(\tau) &= \left(v_0 \cos\left(\sqrt{1-\varsigma^2}\tau\right) + \frac{\dot{u}_0 + v_0\varsigma}{\sqrt{1-\varsigma^2}} \sin\left(\sqrt{1-\varsigma^2}\tau\right) \right) \exp(-\varsigma\tau),\\
\dot{u}(\tau) &= \left(\dot{u}_0 \cos\left(\sqrt{1-\varsigma^2}\tau\right) - \frac{v_0 + \dot{u}_0\varsigma}{\sqrt{1-\varsigma^2}} \sin\left(\sqrt{1-\varsigma^2}\tau\right) \right) \exp(-\varsigma\tau).
\end{aligned} \tag{13}
$$

For sufficiently small perturbations, the motion remains elastic and the origin is asymptotically stable as for classically damped linear oscillators. However, for sufficiently large perturbations, a plastic phase is initiated:

$$
\exists \tau_1 > 0 / \left| v(\tau_1) \right| = 1. \tag{14}
$$

It can be shown that the sign of v during this plastic phase depends on the initial perturbation $\dot{u}_0$. Equation of motion in the plastic regime is given by

$$\begin{aligned}&\ddot{u}+2\varsigma\dot{u}+v=0,\\&|v|=1 \quad \text{with } v(\tau_1)=v_1,\ \dot{u}(\tau_1)=\dot{u}_1,\end{aligned} \tag{15}$$

whose solution is

$$\begin{aligned}&v(\tau)=v_1,\\&\dot{u}(\tau)=\left(\dot{u}_1+v_1\frac{1}{2\varsigma}\right)\exp(-2\varsigma(\tau-\tau_1))-v_1\frac{1}{2\varsigma}.\end{aligned} \tag{16}$$

It is easy to show that $|\dot{u}(\tau)|$ decreases during this plastic phase, and vanishes at time τ_2:

$$\dot{u}(\tau_2)=0 \Rightarrow \tau_2=\tau_1+\frac{1}{2\varsigma}\ln\left(1+2\varsigma\frac{\dot{u}_1}{v_1}\right). \tag{17}$$

At time τ_2, the system goes into an elastic phase and remains in this stage.

$$\begin{aligned}v(\tau)&=\left(v_2\cos\left(\sqrt{1-\varsigma^2}(\tau-\tau_2)\right)+\frac{\dot{u}_2+v_2\varsigma}{\sqrt{1-\varsigma^2}}\sin\left(\sqrt{1-\varsigma^2}(\tau-\tau_2)\right)\right)\exp(-\varsigma(\tau-\tau_2)),\\ \dot{u}(\tau)&=\left(\dot{u}_2\cos\left(\sqrt{1-\varsigma^2}(\tau-\tau_2)\right)-\frac{v_2+\dot{u}_2\varsigma}{\sqrt{1-\varsigma^2}}\sin\left(\sqrt{1-\varsigma^2}(\tau-\tau_2)\right)\right)\exp(-\varsigma(\tau-\tau_2))\\ &\text{with}\quad v(\tau_2)=v_2=v_1,\ \dot{u}(\tau_2)=\dot{u}_2=0.\end{aligned} \tag{18}$$

It means that in both cases (with or without intermediate plastic phase), the origin is asymptotically stable for the damped system $\varsigma > 0$. This is well illustrated in Fig. 3, obtained with a perturbation leading to a plastic transitory phase. This simulation also shows the potential wall analogy for this elastoplastic system. The trajectory that comes from the elastic phase faces the plastic potential wall and dissipates energy until the displacement rate vanishes.

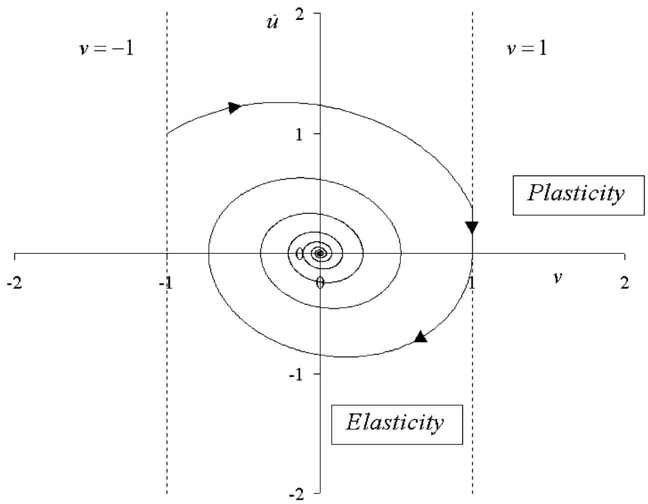

Fig. 3. Free vibrations—asymptotic stability of the origin point $(v, \dot{u}) = (0, 0)$.

3.2. Direct method of Liapounov

It has to be specified that the direct method of Liapounov was initially developed by Liapounov for smooth systems (see for instance La Salle and Lefschetz [30]). Extension of such methodology to non-smooth systems is a recent topic since the pioneer work of Filippov [31] (Shevitz and Paden [32]; Wu and Sepehri [33]; Bourgeot and Brogliato [34]). For the non-smooth system depicted in Eq. (11), a smooth Liapounov function can be chosen as

$$V(\dot{u}, v) = \tfrac{1}{2}\dot{u}^2 + \tfrac{1}{2}v^2. \tag{19}$$

The direct method of Liapounov is based on the calculation of the time derivative of V. Nevertheless, this derivative does not exist in the classical sense at the intersection of the elastic and the plastic states (it only exists almost everywhere). It is possible to show that

$$\frac{\mathrm{d}V(v,\dot{u})}{\mathrm{d}\tau} \in \dot{\tilde{V}} \quad \text{where } \dot{\tilde{V}} = \left(\frac{\partial V}{\partial v}, \frac{\partial V}{\partial \dot{u}}\right)\begin{pmatrix} K[|\dot{u}|[h(1-v) - h(1+v)] + \dot{u}h(1-|v|)] \\ K[-2\varsigma\dot{u} - v] \end{pmatrix}. \tag{20}$$

K is called Filippov's set. This set can be calculated for the Heaviside function

$$K[h(x)] = \mathrm{H}(x) \quad \text{with } \mathrm{H}(x) = \begin{cases} 1 & \text{if } x>0, \\ [0;1] & \text{if } x=0, \\ 0 & \text{if } x<0. \end{cases} \tag{21}$$

$\dot{\tilde{V}}$ can then be simplified as

$$\dot{\tilde{V}} = -2\varsigma\dot{u}^2 + v[|\dot{u}|(H(1-v) - H(1+v)) + \dot{u}(H(1-|v|) - 1)]. \tag{22}$$

The final result is obtained:

$$\begin{aligned} &\hat{E}\text{ state}: \quad \dot{\tilde{V}} = -2\varsigma\dot{u}^2 \leqslant 0, \\ &\hat{P}\text{ state}: \quad \dot{\tilde{V}} = -2\varsigma\dot{u}^2 - 2|\dot{u}|H(0) \leqslant 0. \end{aligned} \tag{23}$$

V is a positive and definite function. Each element of $\dot{\tilde{V}}$ is negative or zero. Then, the origin (0, 0) is stable in the sense of Liapounov (for instance Ref. [32]). Moreover, the non-smooth version of LaSalle's theorem leads to the asymptotic stability in this case. The largest invariance set defined by the vanishing of $\dot{\tilde{V}}$ can be calculated as

$$\dot{u} = 0, \tag{24}$$

$\dot{\tilde{V}}$ vanishes on the $\dot{u} = 0$ axis. Other than the trivial solution $(v, \dot{u}) = (0, 0)$, there is no further solution of Eq. (11) (with $f_0 = 0$) for which $\dot{\tilde{V}}$ vanishes identically. Thus, the trivial solution is asymptotically stable and the domain of attraction is the whole phase space. These results on the asymptotical stability of the origin, are confirmed by the works of Judge and Pratap [35] who treat the more general case of the damped plastic hardening oscillator.

4. Forced vibrations—analysis

4.1. Equations

As for general piecewise linear oscillators, solutions of the periodically forced oscillator Eq. (9) are known explicitly for each state. The initial conditions at the beginning of each state are written by

$$(v(\tau_i), \dot{u}(\tau_i)) = (v_i, \dot{u}_i). \tag{25}$$

The solution to the $\hat{E}$ state can be calculated as

$$v(\tau)=\begin{pmatrix} \cos\left(\sqrt{1-\varsigma^2}(\tau-\tau_i)\right)\left(v_i-f_0\dfrac{(1-\omega^2)\cos(\omega\tau_i)+2\omega\varsigma\sin(\omega\tau_i)}{(1-\omega^2)^2+4\omega^2\varsigma^2}\right)+\sin\left(\sqrt{1-\varsigma^2}(\tau-\tau_i)\right) \\ \left(\dfrac{v_i\varsigma+\dot{u}_i}{\sqrt{1-\varsigma^2}}+f_0\dfrac{-\varsigma(1+\omega^2)\cos(\omega\tau_i)+\omega(1-\omega^2-2\varsigma^2)\sin(\omega\tau_i)}{\sqrt{1-\varsigma^2}\left((1-\omega^2)^2+4\omega^2\varsigma^2\right)}\right) \end{pmatrix}$$

$$\times\, e^{-\varsigma(\tau-\tau_i)}+f_0\frac{(1-\omega^2)\cos(\omega\tau)+2\omega\varsigma\sin(\omega\tau)}{(1-\omega^2)^2+4\varsigma^2\omega^2},$$

$$\dot{u}(\tau)=\begin{pmatrix} \cos\left(\sqrt{1-\varsigma^2}(\tau-\tau_i)\right)\left(\dot{u}_i+f_0\left(\dfrac{-2\omega^2\varsigma\cos(\omega\tau_i)+\omega(1-\omega^2)\sin(\omega\tau_i)}{(1-\omega^2)^2+4\omega^2\varsigma^2}\right)\right)+\sin\left(\sqrt{1-\varsigma^2}(\tau-\tau_i)\right) \\ \left(-\dfrac{v_i+\dot{u}_i\varsigma}{\sqrt{1-\varsigma^2}}+f_0\dfrac{\cos(\omega\tau_i)(2\omega^2\varsigma^2+1-\omega^2)+\sin(\omega\tau_i)(\omega\varsigma(1+\omega^2))}{\sqrt{1-\varsigma^2}\left((1-\omega^2)^2+4\omega^2\varsigma^2\right)}\right) \end{pmatrix}$$

$$\times\, e^{-\varsigma(\tau-\tau_i)}+\omega f_0\frac{2\omega\varsigma\cos(\omega\tau)-(1-\omega^2)\sin(\omega\tau)}{(1-\omega^2)^2+4\varsigma^2\omega^2}. \tag{26}$$

The solution to the $\hat{P}$ state can be calculated as

$$v(\tau)=v_i=\pm 1,$$
$$\dot{u}(\tau)=\left(\dot{u}_i+\frac{v_i}{2\varsigma}-f_0\frac{2\varsigma\cos(\omega\tau_i)+\omega\sin(\omega\tau_i)}{4\varsigma^2+\omega^2}\right)e^{-2\varsigma(\tau-\tau_i)}-\frac{v_i}{2\varsigma}+f_0\frac{2\varsigma\cos(\omega\tau)+\omega\sin(\omega\tau)}{4\varsigma^2+\omega^2}. \tag{27}$$

Piecing together these known solutions is not directly possible however, since the time of flight in each region (each state) cannot be found in closed-form solution for the forced oscillator. The method of locating events is used in the integration process (see Ref. [25] for instance). For the initial conditions specified, the computer solves the crossing time τ_{i+1} using a simple Newton–Raphson method. The nonlinear equation to be solved is given by Eq. (26) when the initial state is elastic:

$$\left|v(\tau_{i+1})\right|=1, \tag{28}$$

whereas the nonlinear equation to be solved is given by Eq. (27) when the initial state is plastic:

$$\dot{u}(\tau_{i+1})=0. \tag{29}$$

The new time τ_{i+1} is used for the equation of motion in the new region encountered. This method is considerably more accurate than usual numerical solutions of ordinary differential equations, the only approximation being made on the calculation of the crossing time.

4.2. Numerical results—shape of the limit cycles

The numerical analysis is performed for strictly positive damping ratios. All trajectories tend towards periodic orbits, which can be viewed as ''limit cycles'' in the $(v,\dot{u})$ space. These ''limit cycles'' do not depend on initial conditions. In reality, the periodic orbits are completely characterized in the $(v,\dot{u},\tau)$ space. Numerical simulations show that these periodic orbits are asymptotically stable for all perturbations. In this sense, the behaviour of the damped elastoplastic oscillator is much simpler than the one of the undamped which does not possess this fundamental property. All these limit cycles are symmetrical cycles (central symmetry with respect to the origin point), as characterized for instance by Luo [36] for other dynamical systems. As a consequence of this symmetry, no incremental collapse can be found for the stable periodic orbits of this damped oscillator (see Ref. [24] for the undamped system). The period of these limit cycles is equal to $2\pi/\omega$. It is interesting to

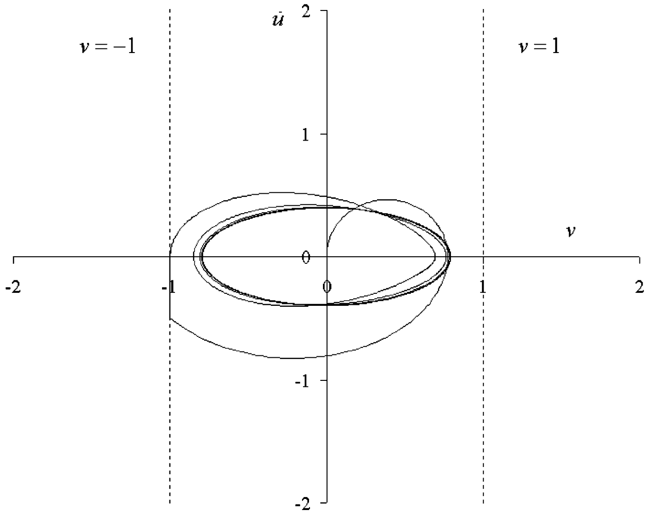

Fig. 4. Elastoplastic shakedown—$(v_0, \dot{u}_0) = (0, 0)$; $\varsigma = 0.1$; $\omega = 0.5$; $f_0 = 0.6$.

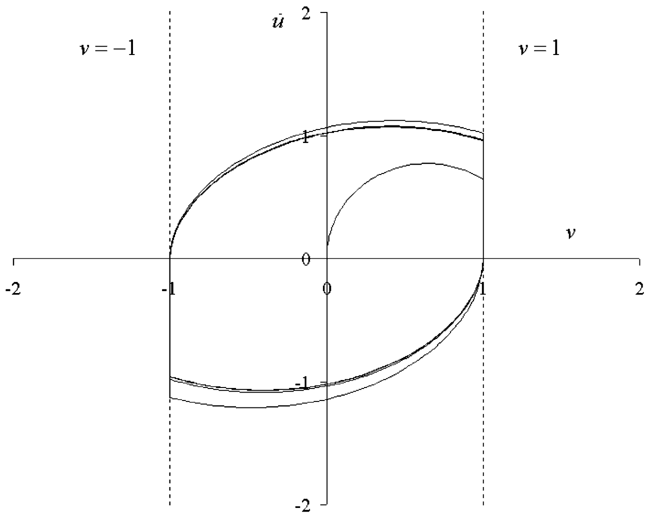

Fig. 5. Alternating plasticity—$(v_0, \dot{u}_0) = (0, 0)$; $\varsigma = 0.1$; $\omega = 0.5$; $f_0 = 1$.

denote that this periodicity result is no more valid for a bilinear elastic oscillator, where the period of the limit cycles has been observed to be different in some cases [37]. The shape of the limit cycles depends on the values of the structural parameters (f_0, ω, ς). Shakedown (elastic stationary evolutions) is described by a smooth limit cycle whereas alternating plasticity is depicted by non-smooth cycles. It means that these two basic phenomena can be differentiated by simple geometrical arguments in the phase space. These two phenomena are illustrated by Fig. 4 (elastoplastic shakedown) and Fig. 5 (alternating plasticity). In both cases, the limit cycles are asymptotically stable and initial conditions do not affect the shape of the limit cycles (see Figs. 6 and 7). However, in both cases, the mean value of the total displacement u strongly depends on initial conditions (Fig. 8). As a consequence, it is confirmed that the initial phase space $(u, u_p, \dot{u})$ cannot be associated with "limit cycles" and is probably not the best choice to analyse the nonlinear dynamics of the plastic system.

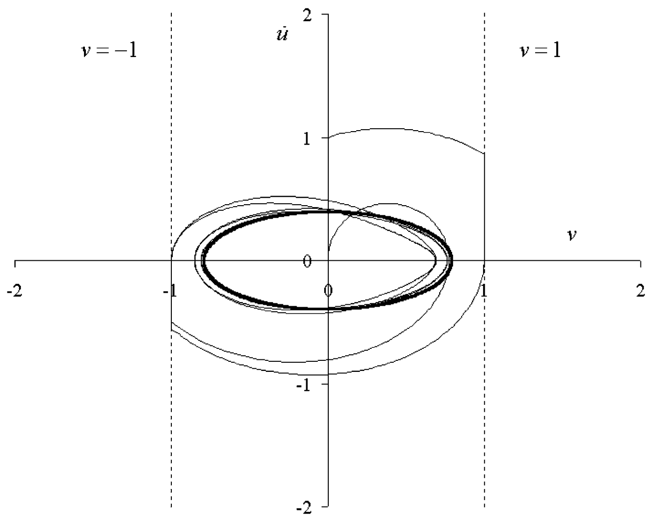

Fig. 6. Elastoplastic shakedown—convergence towards the limit cycle for different initial conditions; $\varsigma = 0.1$; $\omega = 0.5$; $f_0 = 0.6$.

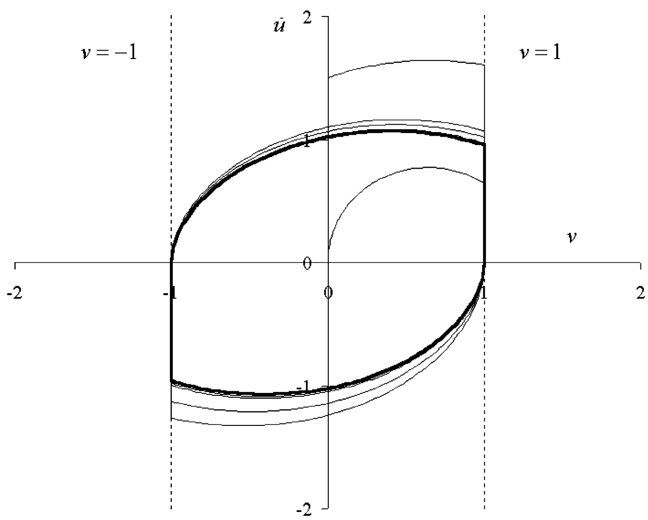

Fig. 7. Alternating plasticity—convergence towards the limit cycle for different initial conditions; $\varsigma = 0.1$; $\omega = 0.5$; $f_0 = 1$.

The equation of the limit cycle in the shakedown area is obtained from Eq. (26) by considering only the stationary term

$$v(\tau) = f_0 \frac{(1-\omega^2)\cos(\omega\tau) + 2\omega\varsigma\,\sin(\omega\tau)}{(1-\omega^2)^2 + 4\varsigma^2\omega^2},$$
$$\dot{u}(\tau) = \omega f_0 \frac{2\omega\varsigma\,\cos(\omega\tau) - (1-\omega^2)\sin(\omega\tau)}{(1-\omega^2)^2 + 4\varsigma^2\omega^2}. \quad (30)$$

This is a centred ellipse whose equation is given by

$$\frac{v^2}{a^2} + \frac{\dot{u}^2}{b^2} = 1 \quad \text{with } a = \frac{f_0}{\sqrt{(1-\omega^2)^2 + 4\omega^2\varsigma^2}} \text{ and } b = a\omega. \quad (31)$$

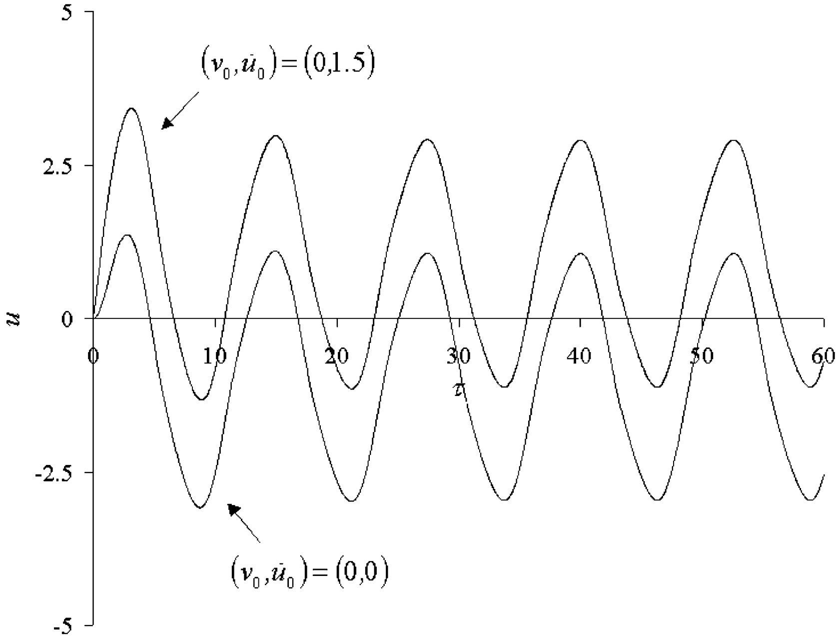

Fig. 8. Influence of initial conditions on the total displacement u; $\varsigma = 0.1$; $\omega = 0.5$; $f_0 = 1$; $u_0 = 0$.

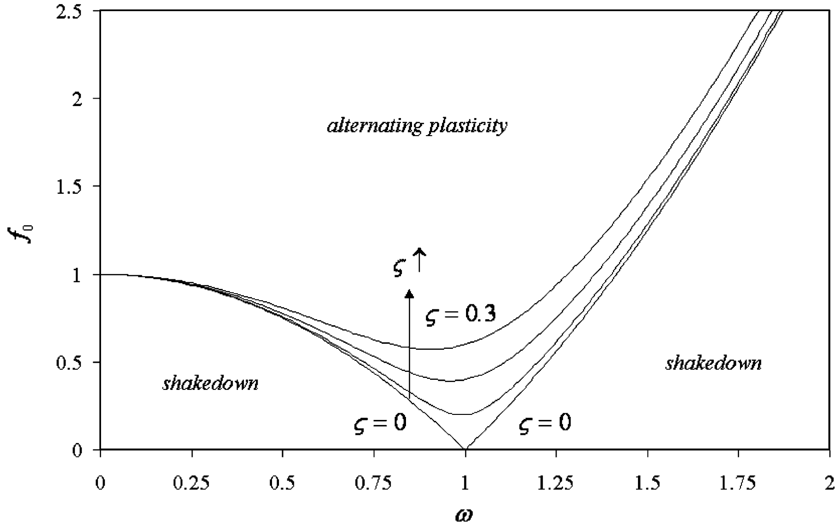

Fig. 9. Boundary between elastoplastic shakedown and incremental collapse in the space (ω, f_0); $\zeta \in \{0, 0.1, 0.2, 0.3\}$.

The equation of the limit cycle at the boundary between shakedown and alternating plasticity is obtained when the ellipse parameter a is equal to 1 in Eq. (31), i.e.

$$f_0 = \sqrt{(1-\omega^2)^2 + 4\omega^2\varsigma^2}. \tag{32}$$

In this case, the equation of the ellipse which is tangent to the axis $|v| = 1$ is reduced to

$$v^2 + \frac{\dot{u}^2}{\omega^2} = 1. \tag{33}$$

This boundary between shakedown and alternating plasticity was already investigated by Liu and Huang [20]. In the case of the undamped system (see also Refs. [14,24]), Eq. (32) can be simplified by

$$f_0 = |1 - \omega^2|. \tag{34}$$

The boundary equation (32) is graphically represented in Fig. 9.

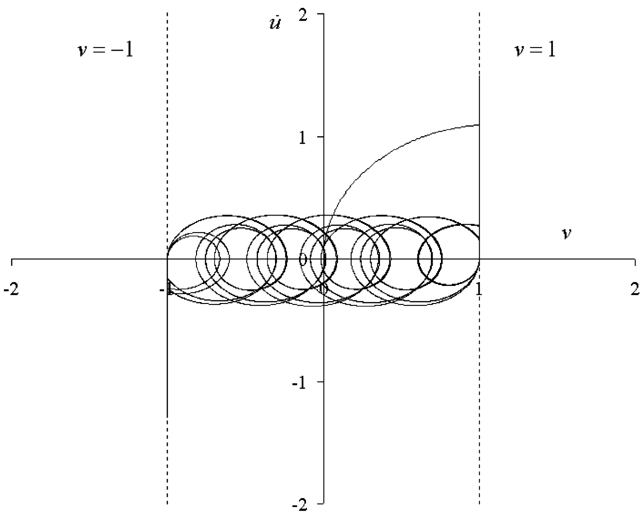

Fig. 10. Convergence towards the non-standard (1, 4)-periodic orbit; $(v_0, \dot{u}_0) = (0, 0)$; $\varsigma = 0.001$; $\omega = 0.05$; $f_0 = 1.1$.

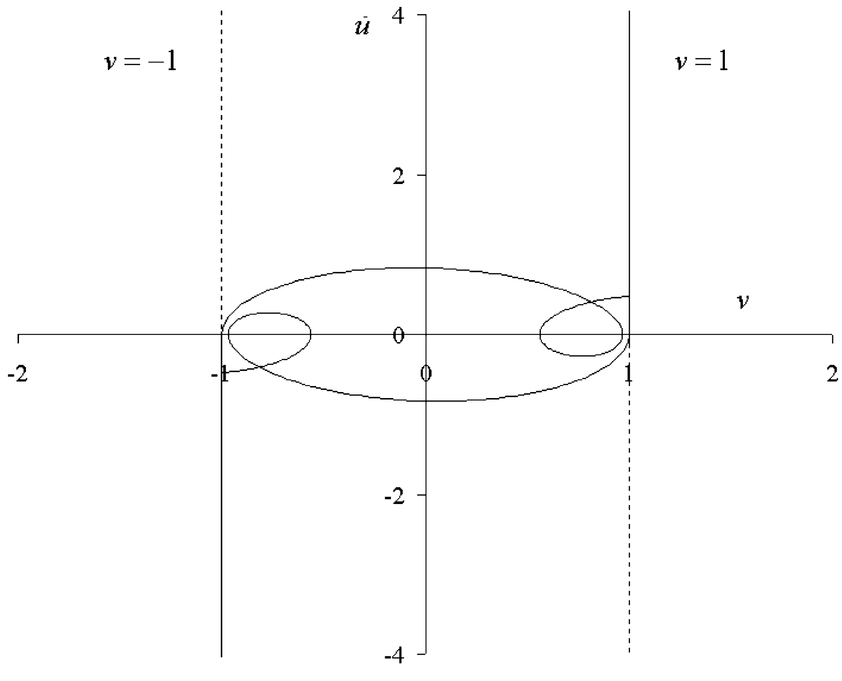

Fig. 11. Non-standard (1, 2)-limit cycle; $\varsigma = 0.1$; $\omega = 0.1$; $f_0 = 2$.

4.3. Numerical results—stability analysis

The periodic alternating plasticity motion can be classified, as for instance by Awrejcewicz and Lamarque [2] for mechanical systems with impacts. We will call (n, k)-periodic solution of period nT with k plastic phases

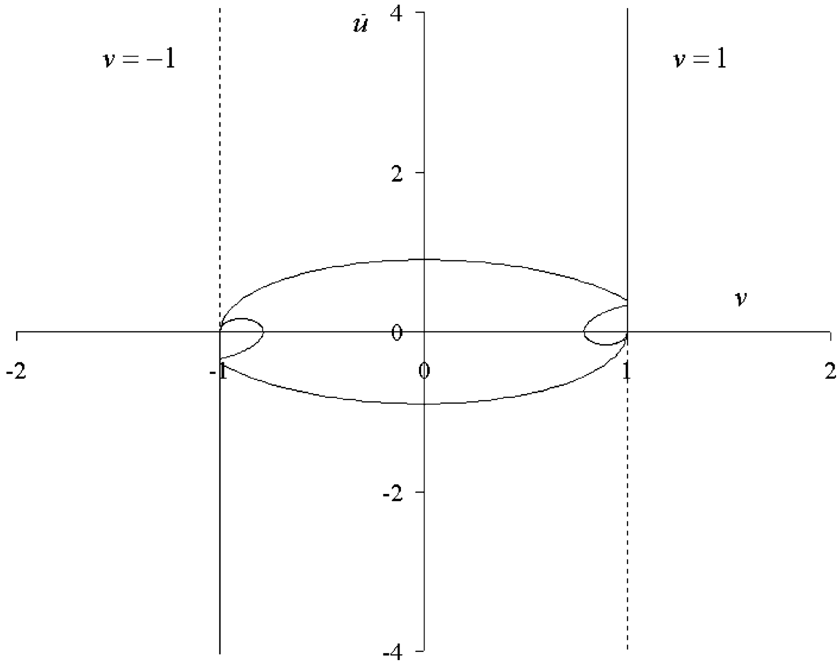

Fig. 12. $(1, 4)$-limit cycle; $\varsigma = 0.1$; $\omega = 0.11$; $f_0 = 2$.

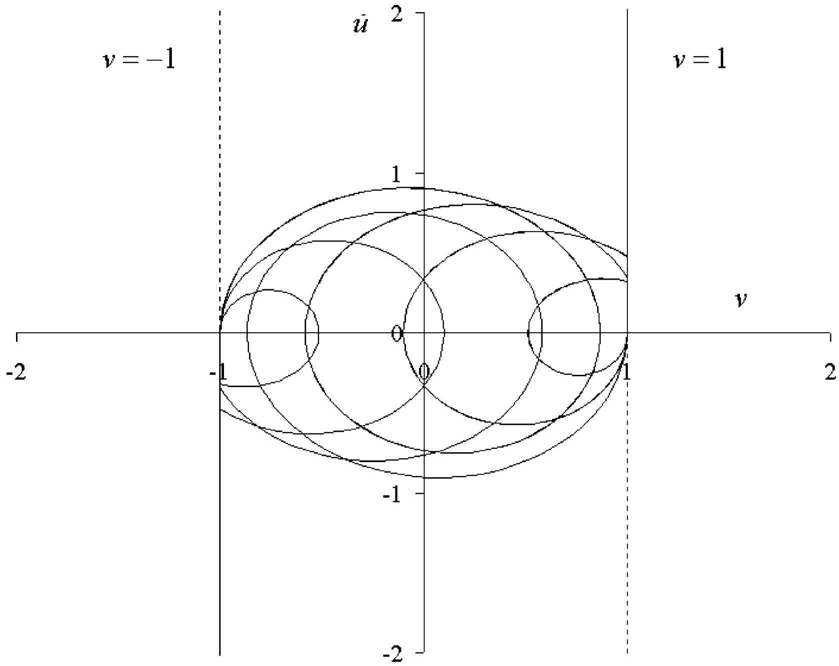

Fig. 13. $(1, 6)$-limit cycle; $\varsigma = 0.02$; $\omega = 0.025$; $f_0 = 2$.

per cycle, where $T = 2\pi/\omega$ is the period of the external excitation. Within this classification, most of the simulations exhibit stable $(1, 2)$-periodic solutions. This is the case for the simulation of Fig. 5. Non-standard 1-periodic solutions can also be observed for small values of the angular frequency ω (see Fig. 10). "Non-standard" here means that higher harmonics are involved during the stationary evolution. The superharmonic

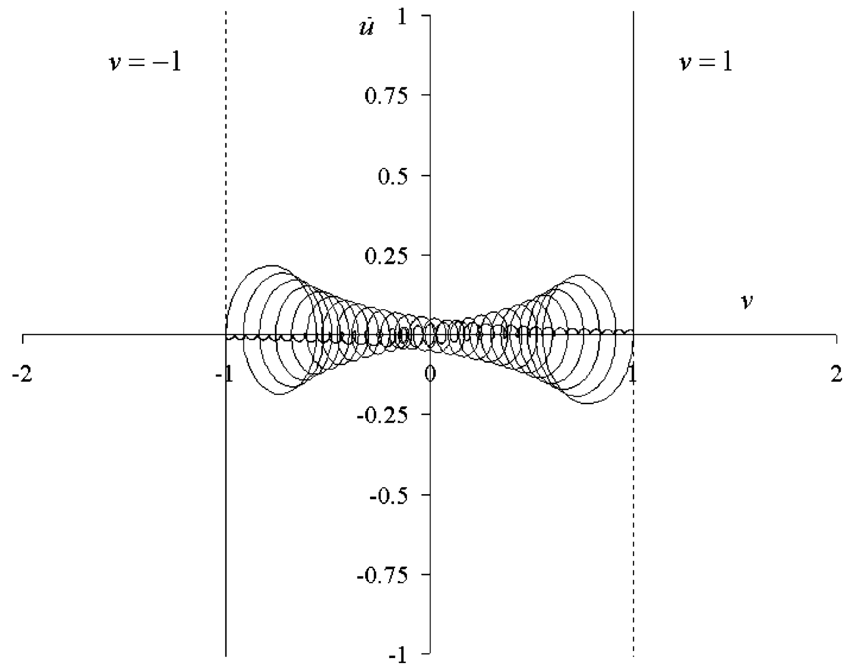

Fig. 14. Non-standard $(1, 2)$-limit cycle; $\varsigma = 0.02$; $\omega = 0.005$; $f_0 = 2$.

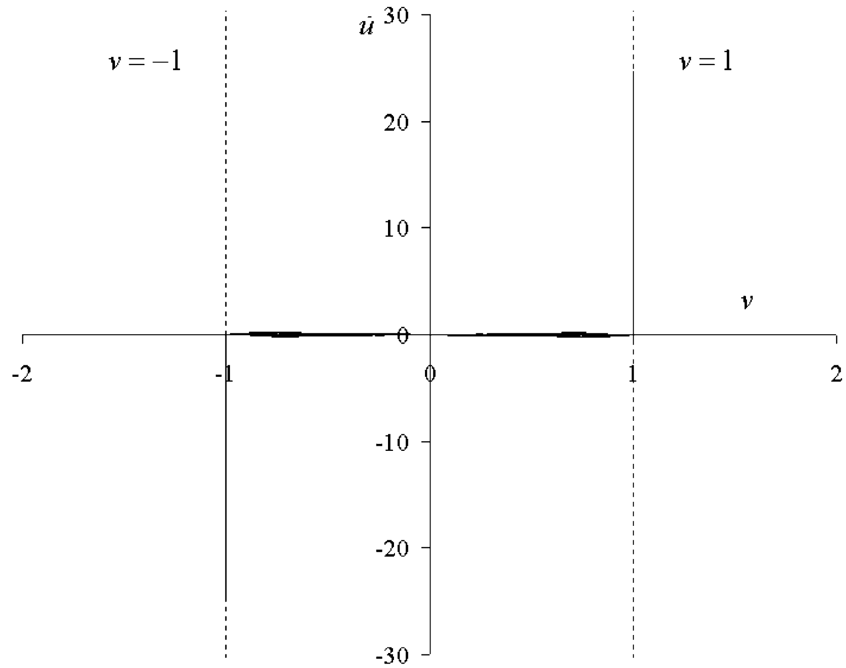

Fig. 15. Non-standard $(1, 2)$-limit cycle; $\varsigma = 0.02$; $\omega = 0.005$; $f_0 = 2$.

resonances occur when the angular frequency is close to $\frac{1}{3},\frac{1}{5},\frac{1}{7},\ldots$ This phenomenon was first mentioned in Ref. [14] or Ref. [15] for the undamped system, or in Ref. [20] for the same damped system. Superharmonic resonance is closely related to the appearance of stable $(1,4)$-periodic solutions or other periodic orbits with more than two plastic phases. The process of the genesis of such orbits is much more clearly explained by considering Figs. 11 and 12. A slight perturbation of the angular frequency ω from 0.1 to 0.11 (other structural parameters ς and f_0 are kept constant) shows the route from $(1,2)$-periodic solution to $(1,4)$-periodic solution. The loop generated by the higher harmonics leads to the $(1,4)$-periodic solution in this case. The number of plastic phases can be greater than two, as shown for instance by the $(1,6)$ periodic solution of Fig. 13. Higher harmonics play a crucial role for small values of the angular frequency ω, as in the case of Fig. 14 ($\omega = 0.005$). The same figure with a change of scale (Fig. 15) indicates that the oscillator behaves like a rigid plastic oscillator in this range, with large values of the displacement rate $\dot{u}$. The general trend of the maximum value of the displacement rate $\dot{u}$ as ω approaches zero is towards an infinite response, although the motion remains periodic for f_0 greater than 1. This characteristic value ($f_0 = 1$) belongs to the boundary between shakedown and alternating plasticity. It corresponds to the static value of the shakedown boundary ($F_0 = F^+$). The case of the undamped elastoplastic oscillator has been already treated in Ref. [24]. A bifurcation boundary separates the alternating plasticity and the shakedown area. This phenomenon will be confirmed later in the paper by considering negative damping ratio values. Elastoplastic shakedown is associated with the dependence on initial conditions for the undamped system. However, most of the presented studies are only based on numerical simulations. The following part is focused on the characterization of these periodic solutions according to their stability properties.

5. Forced vibrations—closed-form solution of some steady-state vibrations

The existence and uniqueness of a periodic and symmetrical $(1,2)$ solution is treated in this part. It is clear however, that these solutions are only a family of solutions of the global system. For instance, unsymmetrical solutions cannot be covered by this fundamental assumption. In the same way, a periodic $(1,4)$ solution cannot be obtained (even if it has been numerically observed for small pulsation values). The time at the beginning of an elastic phase is searched for the stationary periodic $(1,2)$ solution (this time can be evaluated with a constant of $n\pi/\omega$ as the stationary motion is symmetrical). The methodology is based on the fact that the duration in each phase (one elastic phase and one plastic phase) is exactly equal to half a period of the cycle:

$$\tau_2(\tau_0) - \tau_0 = \frac{\pi}{\omega}, \tag{35}$$

where τ_0 is the time at the beginning of the elastic phase following the plastic phase $\hat{P}^-$, τ_1 is the time at the end of this elastic phase and τ_2 is the time at the end of the plastic phase $\hat{P}^+$ (see Fig. 16). The most general method needs the resolution of a nonlinear system of three functions, defined by Eqs. (26) and (27):

$$\begin{aligned}&\hat{E}\text{ state}: \quad v(\tau_0,\tau_1) = 1; \quad \dot{u}_1 = \dot{u}(\tau_0,\tau_1),\\ &\hat{P}^+\text{ state}: \quad \dot{u}(\tau_1,\tau_2,\dot{u}_1) = 0,\\ &\tau_2 - \tau_0 = \frac{\pi}{\omega}.\end{aligned} \tag{36}$$

This system can simply be reduced to two nonlinear functions:

$$\begin{aligned}&\hat{E}\text{ state}: \quad v(\tau_0,\tau_1) = 1; \quad \dot{u}_1 = \dot{u}(\tau_0,\tau_1),\\ &\hat{P}^+\text{ state}: \quad \dot{u}\left(\tau_1,\tau_0 + \frac{\pi}{\omega},\dot{u}_1\right) = 0,\end{aligned} \tag{37}$$

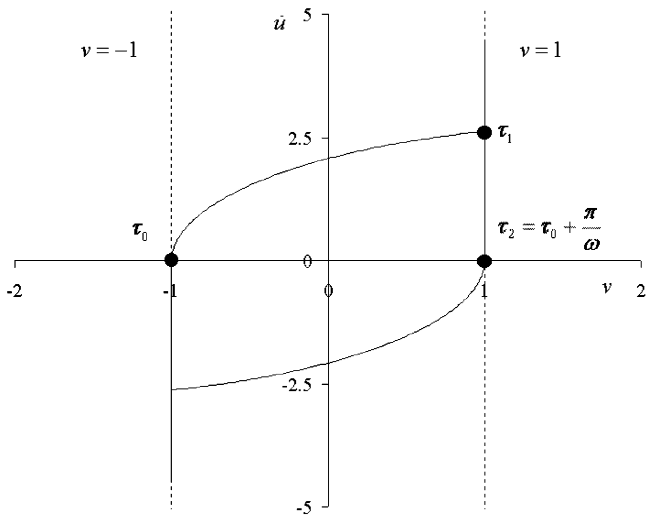

Fig. 16. Definition of the time-parameters of the limit cycle.

which can be expressed by

$$\begin{gathered}\left(\begin{array}{l}\cos\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)*\left(-1-f_0\dfrac{(1-\omega^2)\cos(\omega\tau_0)+2\omega\varsigma\,\sin(\omega\tau_0)}{(1-\omega^2)^2+4\omega^2\varsigma^2}\right)+\sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\\ \left(\dfrac{-\varsigma}{\sqrt{1-\varsigma^2}}+f_0\dfrac{-\varsigma(1+\omega^2)\cos(\omega\tau_0)+\omega(1-\omega^2-2\varsigma^2)\sin(\omega\tau_0)}{\sqrt{1-\varsigma^2}\left((1-\omega^2)^2+4\omega^2\varsigma^2\right)}\right)\end{array}\right)\\ \times\,\mathrm{e}^{-\varsigma(\tau_1-\tau_0)}+f_0\dfrac{(1-\omega^2)\cos(\omega\tau_1)+2\omega\varsigma\,\sin(\omega\tau_1)}{(1-\omega^2)^2+4\varsigma^2\omega^2}=1,\\ \left(\dot{u}_1(\tau_0,\tau_1)+\dfrac{1}{2\varsigma}-f_0\dfrac{2\varsigma\,\cos(\omega\tau_1)+\omega\,\sin(\omega\tau_1)}{4\varsigma^2+\omega^2}\right)\mathrm{e}^{-2\varsigma\left(\tau_0+(\pi/\omega)-\tau_1\right)}-\dfrac{1}{2\varsigma}\\ -f_0\dfrac{2\varsigma\,\cos(\omega\tau_0)+\omega\,\sin(\omega\tau_0)}{4\varsigma^2+\omega^2}=0\end{gathered}$$

with

$$\begin{gathered}\dot{u}_1=\left(\begin{array}{l}\cos\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)f_0\left(\dfrac{-2\omega^2\varsigma\,\cos(\omega\tau_0)+\omega(1-\omega^2)\sin(\omega\tau_0)}{(1-\omega^2)^2+4\omega^2\varsigma^2}\right)+\sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\\ \left(\dfrac{1}{\sqrt{1-\varsigma^2}}+f_0\dfrac{\cos(\omega\tau_0)(2\omega^2\varsigma^2+1-\omega^2)+\sin(\omega\tau_0)(\omega\varsigma(1+\omega^2))}{\sqrt{1-\varsigma^2}\left((1-\omega^2)^2+4\omega^2\varsigma^2\right)}\right)\end{array}\right)\\ \times\,\mathrm{e}^{-\varsigma(\tau_1-\tau_0)}+\omega f_0\dfrac{2\omega\varsigma\,\cos(\omega\tau_1)-(1-\omega^2)\sin(\omega\tau_1)}{(1-\omega^2)^2+4\varsigma^2\omega^2}.\end{gathered}\tag{38}$$

It is possible to write the complex system Eq. (38) as

$$\begin{aligned}\hat{A}\,\cos(\omega\tau_0)+\hat{B}\,\sin(\omega\tau_0)&=\hat{C},\\ \hat{D}\,\cos(\omega\tau_0)+\hat{E}\,\sin(\omega\tau_0)&=\hat{F},\end{aligned}\tag{39}$$

where the coefficients $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\hat{E}$ and $\hat{F}$ depend on the structural parameters (ω, ς, f_0) but also on the time variable $\overline{y} = \tau_1 - \tau_0$ (see Appendix A). As remarked by Liu and Huang [20], this system can be merged into one single nonlinear equation of $\overline{y}$:

$$\left(\hat{C}\hat{D} - \hat{A}\hat{F}\right)^2 + \left(\hat{C}\hat{E} - \hat{B}\hat{F}\right)^2 = \left(\hat{A}\hat{E} - \hat{B}\hat{D}\right)^2. \tag{40}$$

The time τ_0 follows from the computation of the trigonometrical equation

$$\cos(\omega\tau_0) = \frac{\hat{C}\hat{E} - \hat{B}\hat{F}}{\hat{A}\hat{E} - \hat{B}\hat{D}},$$
$$\sin(\omega\tau_0) = \frac{-\hat{C}\hat{D} + \hat{A}\hat{F}}{\hat{A}\hat{E} - \hat{B}\hat{D}} \quad \text{if} \quad \hat{A}\hat{E} - \hat{B}\hat{D} \neq 0. \tag{41}$$

The solutions y ($y = \omega\overline{y}$) of Eq. (40) are plotted in Fig. 17 for $\varsigma = 0.02$ and $f_0 = 2$. The results are the same as the ones of Liu and Huang [20]. However, the interpretation of this curve is quite different from

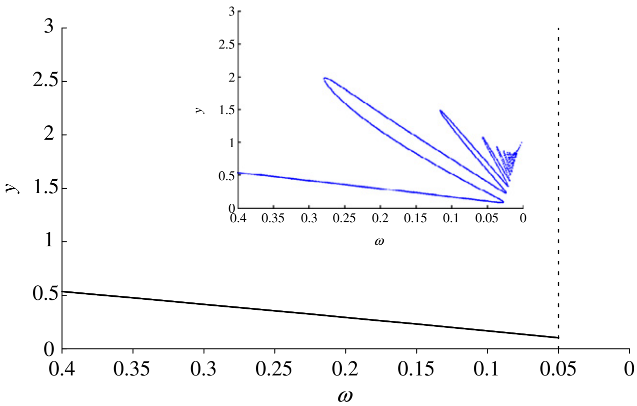

Fig. 17. Roots of y versus ω; $y = \omega\overline{y}$; $\varsigma = 0.02$; $f_0 = 2$ (comparison with the necessary condition of Liu and Huang [20]).

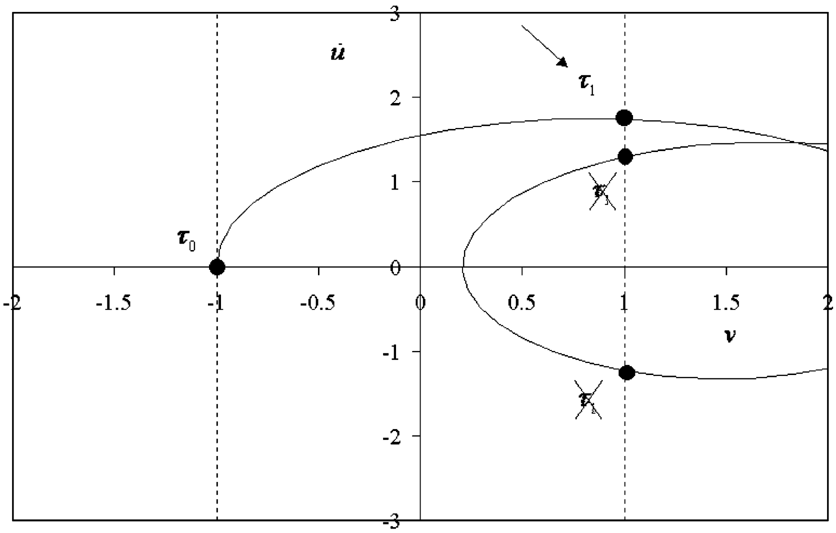

Fig. 18. Selection of admissible values of τ_1.

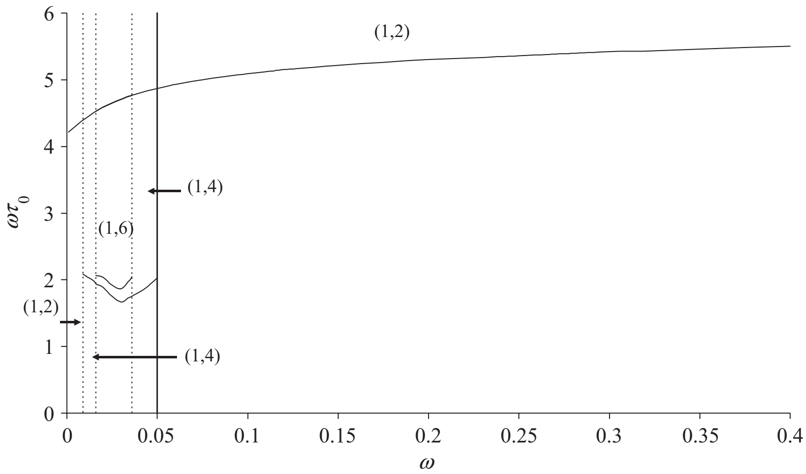

Fig. 19. Bifurcation diagram; $\varsigma = 0.02$; $f_0 = 2$.

the conclusion of [20]. Eq. (40) contains the time solution of the symmetrical (1, 2) orbit but all solutions of Eq. (40) are not necessarily solution of the dynamical problem. In other words, Eq. (40) is only a necessary condition to be fulfilled but not a sufficient condition. One also has to verify the inequalities:

$$\hat{E}\text{ state}: \ \forall\tau \in [\tau_0, \tau_1], \ \ |v(\tau)| \leqslant 1 \quad \text{and} \quad \hat{P}^{+}\text{ state}: \ \forall\tau \in [\tau_1, \tau_2], \ \dot{u}(\tau) \geqslant 0. \tag{42}$$

It can be shown in this case that there is at most only one solution for parameters chosen in Fig. 17. Other solutions violate the conditions of Eq. (42), as shown by Fig. 18 for instance. In fact, there is exactly one solution for ω greater than 0.05 and the parameters chosen for Fig. 17. This solution is the smallest one of the solutions of Eq. (40). The bifurcation diagram of Fig. 19 numerically confirms this analytical conclusion, and also highlights symmetrical (1, 4) and (1, 6) orbits for ω lesser than 0.05.

6. Stability analysis of the symmetrical (1, 2) periodic orbit

The stability analysis of the symmetrical (1, 2) periodic orbit is based on a perturbation technique introduced by Masri and Caughey (1966) [13]:

$$\begin{pmatrix} \Delta\tau_0 \\ \Delta v_0 \\ \Delta\dot{u}_0 \end{pmatrix} \to \begin{pmatrix} \Delta\tau_1 \\ \Delta v_1 \\ \Delta\dot{u}_1 \end{pmatrix} \to \begin{pmatrix} \Delta\tau_2 \\ \Delta v_2 \\ \Delta\dot{u}_2 \end{pmatrix} \quad \text{with } \Delta v_0 = \Delta\dot{u}_0 = \Delta v_1 = \Delta v_2 = \Delta\dot{u}_2 = 0. \tag{43}$$

The matrix **A** and **B** can be introduced for the propagation of errors:

$$\begin{pmatrix} \Delta\tau_1 \\ \Delta v_1 \\ \Delta\dot{u}_1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} \Delta\tau_0 \\ \Delta v_0 \\ \Delta\dot{u}_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Delta\tau_2 \\ \Delta v_2 \\ \Delta\dot{u}_2 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} \begin{pmatrix} \Delta\tau_1 \\ \Delta v_1 \\ \Delta\dot{u}_1 \end{pmatrix}. \tag{44}$$

Eqs. (43) and (44) lead to the scalar equation:

$$\Delta\tau_2 = R\Delta\tau_0 \quad \text{with } R = A_{11}B_{11} + A_{31}B_{13}. \tag{45}$$

The value of R determines the stability of the solution [13]. The symmetric (1, 2) solution is asymptotically stable if the modulus of R is less than unity. If the modulus of R is greater than one, the solution is unstable.

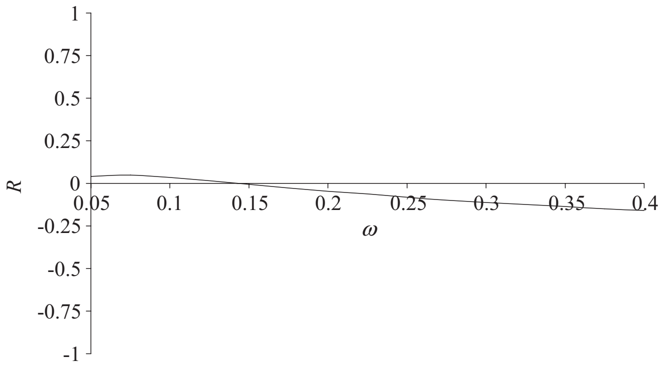

Fig. 20. $|R|<1$; Asymptotic stability of the symmetrical $(1,2)$ orbit; $\varsigma = 0.02$; $f_0 = 2$.

A bifurcation may occur when the modulus takes the value of unity. The coefficients that appear in the calculation of R are given in Appendix B. R can be finally simplified in

$$R = \mathrm{e}^{-\varsigma\left(\tau_0-\tau_1+(2\pi/\omega)\right)}\left[-\cos\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)+\frac{\varsigma}{\sqrt{1-\varsigma^2}}\sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\right]. \tag{46}$$

In the case of the undamped system, this coefficient is reduced to

$$\left|R(\varsigma=0)\right| = \left|\cos(\tau_1-\tau_0)\right| \leqslant 1. \tag{47}$$

Generally, $|R|$ is strictly less than unity ($\tau_1 \neq \tau_0 + n\pi$). This new result (obtained from a rigorous proof for this non-smooth mechanical system) means that the symmetrical $(1,2)$ periodic orbit (when it exists) is always asymptotically stable for the undamped system. It can be numerically checked that Eq. (46) leads to the same conclusion for the damped system. The variation of R with respect to ω is given in Fig. 20 which confirms the asymptotic stability property.

7. Equivalent damping ratio and negative damping

There is a specific and large literature devoted to the determination of an equivalent damping ratio ([38–40] or [41]). The numerous ways to get frequency and amplitude matching are the resonant amplitude method, dynamic stiffness method, dynamic mass method … (see the recent paper [41]). The equivalent damping ratio may be defined such that the associated linear damped system and the initial hysteretic one have the same frequency response curves [41]. All these methods are fitting methods that approximate the equivalent damping ratio from the unknown response of the elastoplastic oscillator. Another point of view consists in determining the equivalent damping ratio directly from the values of the structural parameters (f_0, ω, ς). In the alternating plasticity region, the dissipation could be quantified by adding negative damping such that the dynamical system bifurcates from its 1-periodic orbit. For the undamped system ($\varsigma = 0$), a possible dynamical definition of this parameter would be

$$\begin{gathered}\text{Find } \varsigma_{\text{eq}} \geqslant 0 \text{ for } f_0 \geqslant \left|1-\omega^2\right| \text{ such that the 1-periodic orbit of the dynamical}\\ \text{system (9) associated to the damping ratio } -\varsigma_{\text{eq}} \text{ is no more stable.}\end{gathered} \tag{48}$$

As a consequence, the resolution of problem (48) needs to solve system (9) with a negative damping ratio. We present some fundamental results for this particular system. However, it is not the purpose of the paper to investigate all the complexities of such quite artificial systems, even if negative damping may arise from aerodynamic forces [42]. Very interesting phenomena such as the grazing bifurcation have been observed. Fig. 21 represents a typical example of a phase portrait and Fig. 22, obtained with the same parameters, shows

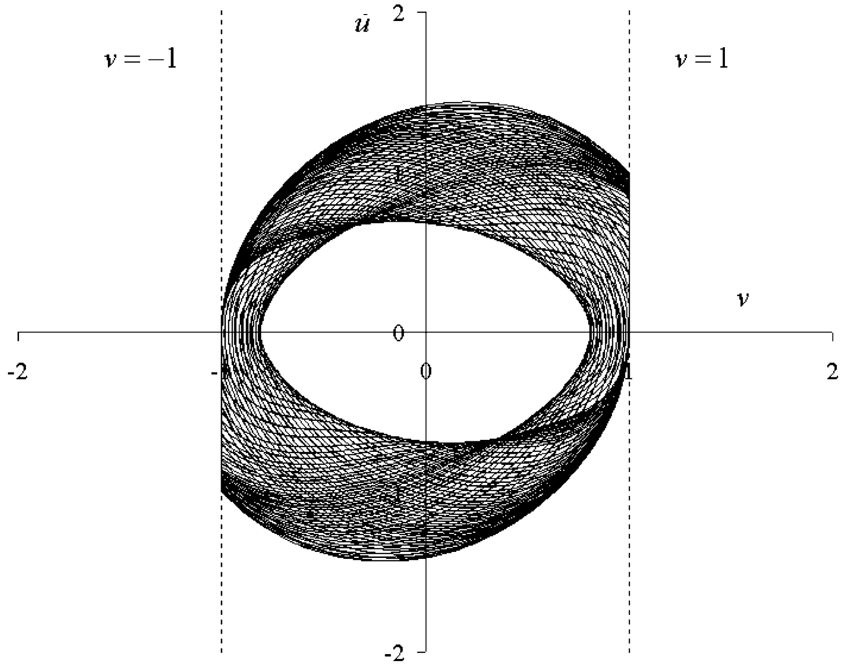

Fig. 21. Quasi-periodic orbits for negative damping values; $\varsigma = -0.05$; $\omega = 2$; $f_0 = 0.5$; phase portraits with $(v_0, \dot{u}_0) = (0, 0)$.

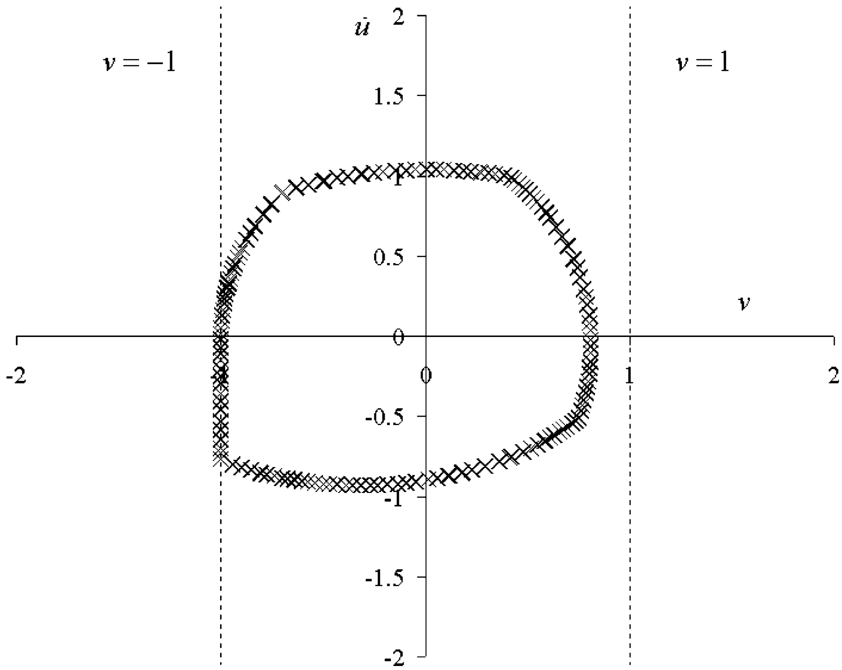

Fig. 22. Quasi-periodic orbits for negative damping values; $\varsigma = -0.05$; $\omega = 2$; $f_0 = 0.5$; Poincaré map with $(v_0, \dot{u}_0) = (0, 0)$; $\tau \equiv 0[2\pi/\omega]$.

the quasi-periodic nature of the dynamics in this range using a Poincaré map. Unsymmetrical limit cycles have also been detected (Fig. 23). This unsymmetrical motion (similar to results of Luo [43] for other mechanical systems) is associated to incremental collapse for this elastoplastic oscillator. Divergence evolution can also be noticed for this negative damped system (Fig. 24). The period-doubling bifurcation has also been observed at

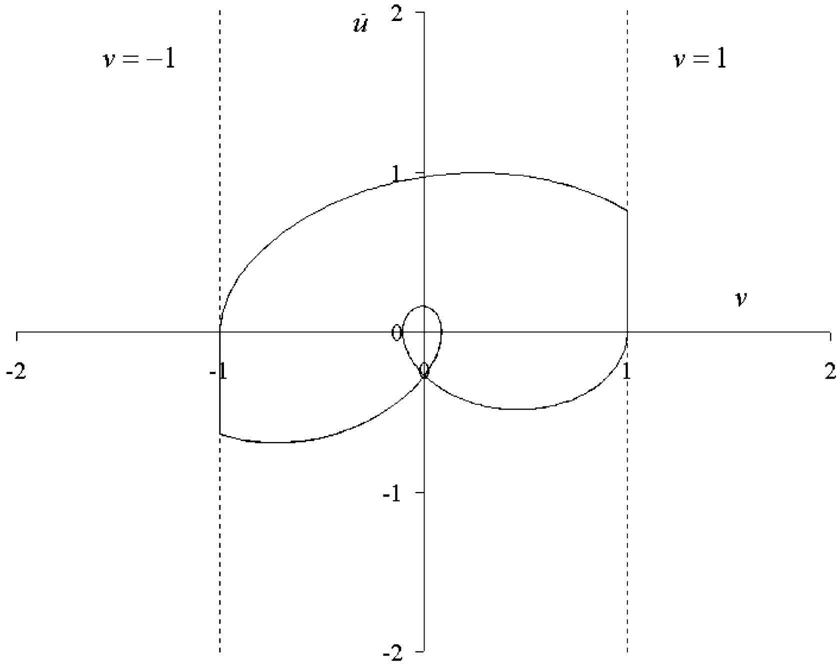

Fig. 23. Unsymmetrical limit cycle; $\varsigma = -0.1$; $\omega = 0.45$; $f_0 = 0.6$.

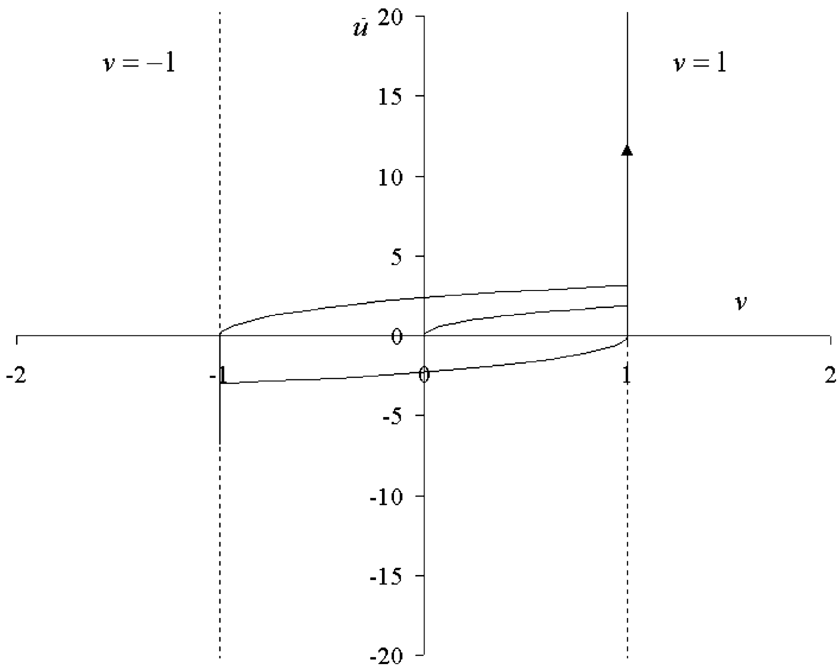

Fig. 24. Divergence evolution; $\varsigma = -0.1$; $\omega = 0.25$; $f_0 = 2$ with $(v_0, \dot{u}_0) = (0, 0)$.

the boundary along the elastoplastic shakedown region (Fig. 25). It has to be mentioned that Shaw [42] obtained very similar results for the dynamic response of a system with dry friction, and with negative viscous damping. The period-doubling bifurcation detected via the Poincaré map is illustrated in Fig. 26. Finally, the

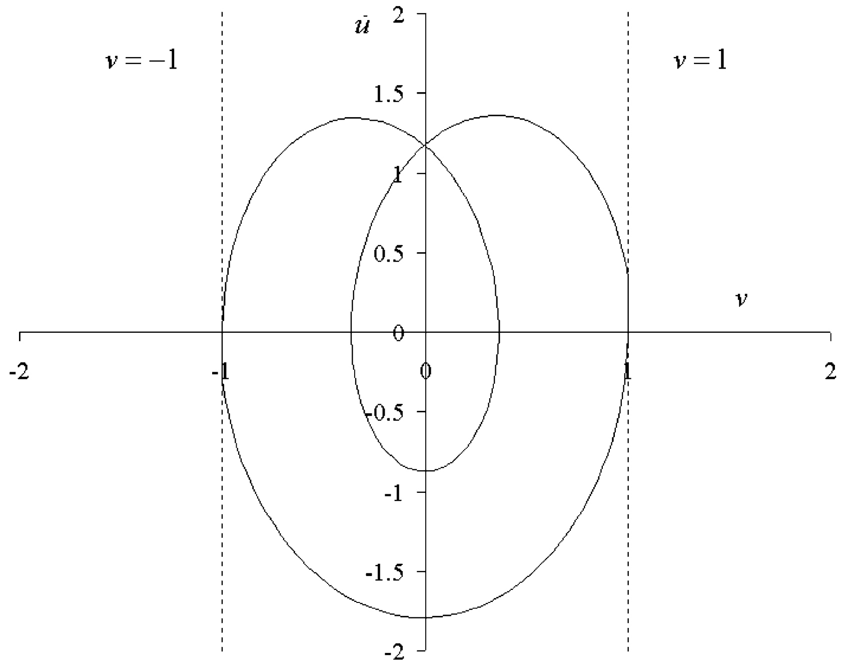

Fig. 25. Stable (2, 2) periodic orbit; $\varsigma = -0.01$; $\omega = 2$; $f_0 = 2$ with $(v_0, \dot{u}_0) = (0, 0)$.

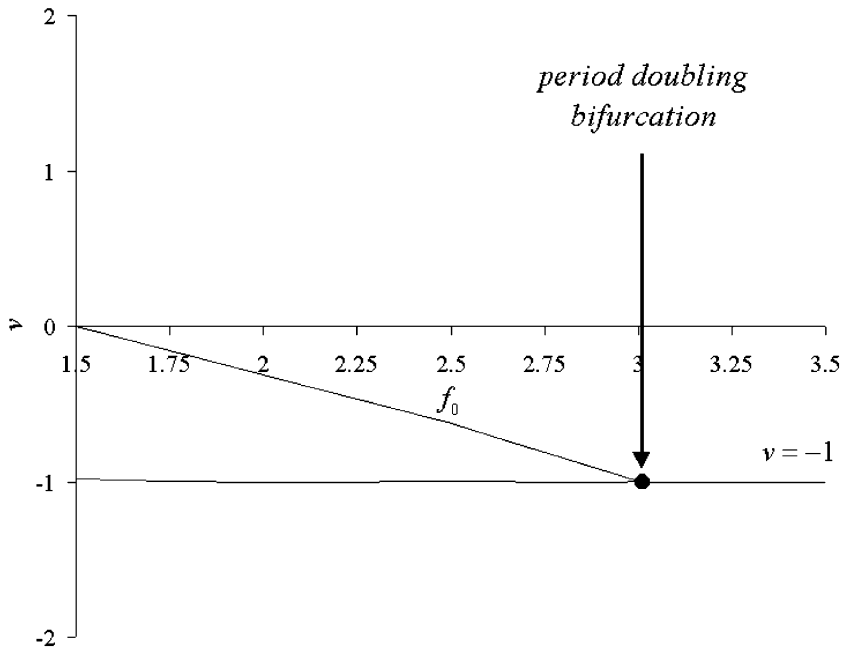

Fig. 26. Period-doubling bifurcation; $\varsigma = -0.05$; $\omega = 2$; $\tau \equiv 0[2\pi/\omega]$.

stability boundary of the period-1 orbit is represented in a multiparameter space (ς, f_0) for constant angular frequency ω (Fig. 27). It can be observed that the boundary of the stability domain is singular. The point which delimitates elastoplastic shakedown and alternating collapse for the undamped system plays a crucial role in this two-dimensional space. The boundary between elastoplastic shakedown and alternating plasticity

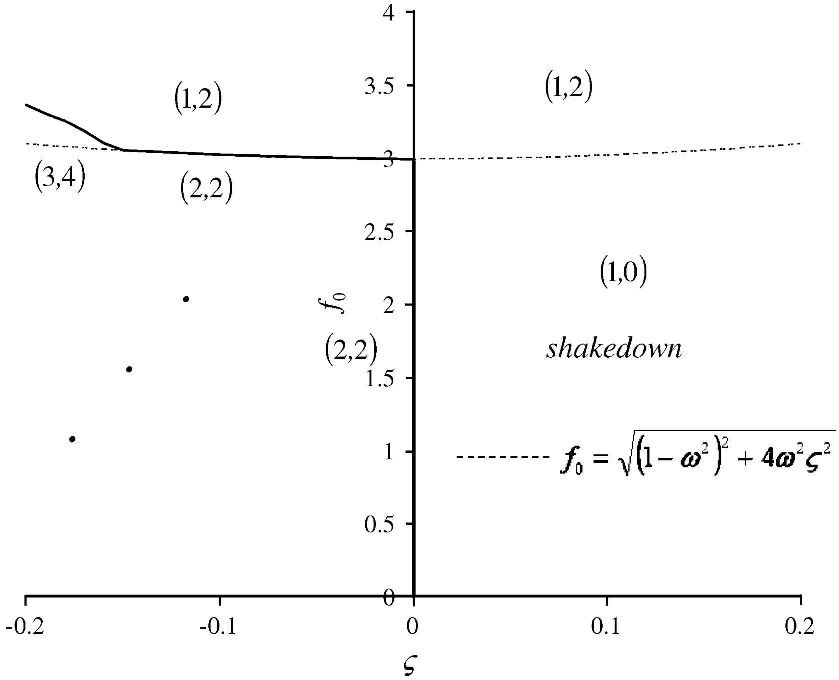

Fig. 27. Stability boundary of the period-1 orbit and its singularities; $\omega = 2$.

corresponds to a bifurcation boundary for the undamped system (period-doubling bifurcation). Moreover, it is numerically observed that equation of the boundary of the elastoplastic shakedown—alternating plasticity domains (see Eq. (32)) seems also to be valid in the case of negative damping values when a period doubling bifurcation occurs (see the stability domain of Fig. 27 for negative damping values):

$$f_0 = \sqrt{(1-\omega^2)^2 + 4\omega^2\varsigma_{eq}^2}. \tag{49}$$

Eq. (49) leads to the identification of the equivalent damping ratio:

$$\varsigma_{eq} = \sqrt{\frac{f_0^2 - (1-\omega^2)^2}{4\omega^2}} \quad \text{for } f_0 \geqslant |1-\omega^2|. \tag{50}$$

This very simple evaluation also coincides with the positive damping to be added to the undamped elastoplastic oscillator, for a given set of parameters (f_0, ω) in the alternating plasticity domain, in order to lie on the boundary of the elastoplastic shakedown domain. However, it appears clearly that Eq. (49) (represented by broken lines in Fig. 27) is no longer valid in the case of the (1, 2)–(3, 4) bifurcation (Fig. 27). Eq. (50) may be generalized to a damped system (with ς damped ratio):

$$\varsigma + \varsigma_{eq} = \sqrt{\frac{f_0^2 - (1-\omega^2)^2}{4\omega^2}} \quad \text{for } f_0^2 \geqslant (1-\omega^2)^2 + 4\omega^2\varsigma^2. \tag{51}$$

8. Conclusions

This paper is devoted to the stability and the dynamics of a harmonically excited damped elastic-perfectly plastic oscillator. The hysteretical system is written as a non-smooth forced system. It is shown that the dimension of the phase space can be reduced using adapted variables. Free vibrations of such a system are then considered for the damped system, which is a non-smooth autonomous system. The direct method of

Liapounov extended to non-smooth systems [32–34] can be applied. The asymptotic stability of the origin is proven in the new phase space. It can be remarked that the non-smooth character of most inelastic behaviours needs the use of some specific tools to investigate the Liapounov stability of inelastic systems. The application of the extended direct method of Liapounov to plastic systems has not been reported in the literature to the authors knowledge.

The forced vibrations of such an oscillator are treated by numerical approach, by using the time locating techniques. The stability of the limit cycles is analytically investigated with a perturbation approach. It has been rigorously proven that the limit cycles are asymptotically stable in case of alternating plasticity. The boundary between elastoplastic shakedown and alternating plasticity is given in closed form. It is shown that this boundary corresponds to a bifurcation boundary for the undamped system (period-doubling bifurcation). This result confirms previous results numerically reported by Challamel [24]. Finally, the equivalent damping of this hysteretic system is characterised from dynamical properties.

This study may be enriched by integrating plastic hardening (isotropic, kinematic or mixed hardening), as studied by Savi and Pacheco [27]. Plastic softening can also be introduced, as considered by Challamel and Pijaudier-Cabot [29] for the free inelastic oscillator. In this case, divergence evolutions can be observed, depending on initial conditions. Damage systems can be also classified as piece-wise linear oscillators (see for instance Challamel and Pijaudier-Cabot [28]). The coupling between plasticity and damage could be also interesting to investigate. However, we have in mind that in any cases (hardening plasticity, softening plasticity, damage or coupled systems), the dimension of the phase space could not be reduced to a map as for the perfect-plastic system. Theoretical results on stability will certainly be more difficult to obtain.

Appendix A. Time characteristics of the symmetric (1, 2) solutions

System (A.1) characterizes the time parameters of the symmetrical (1, 2) periodical solution:

$$\begin{gathered}\hat{A}\cos(\omega\tau_0)+\hat{B}\sin(\omega\tau_0)=\hat{C},\\ \hat{D}\cos(\omega\tau_0)+\hat{E}\sin(\omega\tau_0)=\hat{F}\quad\text{and}\quad\overline{y}=\tau_1-\tau_0.\end{gathered}\tag{A.1}$$

The constants of system (A.1) denoted by $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\hat{E}$ and $\hat{F}$ are given below:

$$\hat{A}=\frac{f_0}{(1-\omega^2)^2+4\omega^2\varsigma^2}\left[\begin{array}{l}-(1-\omega^2)\mathrm{e}^{-\varsigma\overline{y}}\cos\left(\sqrt{1-\varsigma^2}\overline{y}\right)-\dfrac{\varsigma}{\sqrt{1-\varsigma^2}}(1+\omega^2)\mathrm{e}^{-\varsigma\overline{y}}\sin\left(\sqrt{1-\varsigma^2}\overline{y}\right)\\ +(1-\omega^2)\cos(\omega\overline{y})+2\omega\varsigma\sin(\omega\overline{y})\end{array}\right],\tag{A.2}$$

$$\hat{B}=\frac{f_0}{(1-\omega^2)^2+4\omega^2\varsigma^2}\left[\begin{array}{l}-2\omega\varsigma\mathrm{e}^{-\varsigma\overline{y}}\cos\left(\sqrt{1-\varsigma^2}\overline{y}\right)+\dfrac{\omega(1-\omega^2-2\varsigma^2)}{\sqrt{1-\varsigma^2}}\mathrm{e}^{-\varsigma\overline{y}}\sin\left(\sqrt{1-\varsigma^2}\overline{y}\right)\\ -(1-\omega^2)\sin(\omega\overline{y})+2\omega\varsigma\cos(\omega\overline{y})\end{array}\right],\tag{A.3}$$

$$\hat{C}=1+\mathrm{e}^{-\varsigma\overline{y}}\cos\left(\sqrt{1-\varsigma^2}\overline{y}\right)+\frac{\varsigma}{\sqrt{1-\varsigma^2}}\mathrm{e}^{-\varsigma\overline{y}}\sin\left(\sqrt{1-\varsigma^2}\overline{y}\right),\tag{A.4}$$

$$\begin{aligned}\hat{D}=&-\frac{f_0}{4\varsigma^2+\omega^2}\left[2\varsigma+(2\varsigma\cos(\omega\overline{y})+\omega\sin(\omega\overline{y}))\mathrm{e}^{-2\varsigma((\pi/\omega)-\overline{y})}\right]+\frac{f_0}{(1-\omega^2)^2+4\omega^2\varsigma^2}\\ &\left[\begin{array}{l}-2\omega^2\varsigma e^{-\varsigma((2\pi/\omega)-\overline{y})}\cos\left(\sqrt{1-\varsigma^2}\overline{y}\right)+\dfrac{2\omega^2\varsigma^2+1-\omega^2}{\sqrt{1-\varsigma^2}}\mathrm{e}^{-\varsigma((2\pi/\omega)-\overline{y})}\sin\left(\sqrt{1-\varsigma^2}\overline{y}\right)\\ +2\omega^2\varsigma\mathrm{e}^{-2\varsigma((\pi/\omega)-\overline{y})}\cos(\omega\overline{y})-\omega(1-\omega^2)\mathrm{e}^{-2\varsigma((\pi/\omega)-\overline{y})}\sin(\omega\overline{y})\end{array}\right],\end{aligned}\tag{A.5}$$

$$\hat{E} = -\frac{f_0}{4\varsigma^2+\omega^2}\left[\omega + (\omega\cos(\omega\bar{y}) - 2\varsigma\sin(\omega\bar{y}))\,\mathrm{e}^{-2\varsigma((\pi/\omega)-\bar{y})}\right] + \frac{f_0}{(1-\omega^2)^2+4\omega^2\varsigma^2}$$
$$\left[\begin{array}{l}\omega(1-\omega^2)\mathrm{e}^{-\varsigma((2\pi/\omega)-\bar{y})}\cos\left(\sqrt{1-\varsigma^2}\bar{y}\right) + \dfrac{\omega\varsigma(1+\omega^2)}{\sqrt{1-\varsigma^2}}\mathrm{e}^{-\varsigma((2\pi/\omega)-\bar{y})}\sin\left(\sqrt{1-\varsigma^2}\bar{y}\right)\\ -2\omega^2\varsigma\mathrm{e}^{-2\varsigma((\pi/\omega)-\bar{y})}\sin(\omega\bar{y}) - \omega(1-\omega^2)\mathrm{e}^{-2\varsigma((\pi/\omega)-\bar{y})}\cos(\omega\bar{y})\end{array}\right], \quad \text{(A.6)}$$

$$\hat{F} = \frac{1}{2\varsigma}\left(1-\mathrm{e}^{-2\varsigma((\pi/\omega)-\bar{y})}\right) - \mathrm{e}^{-\varsigma((2\pi/\omega)-\bar{y})}\,\frac{\sin\left(\sqrt{1-\varsigma^2}\bar{y}\right)}{\sqrt{1-\varsigma^2}}. \quad \text{(A.7)}$$

This system leads to the solution of Liu and Huang [20].

Appendix B. Determination of the coefficients for the stability analysis

The first coefficient to be determined is A_{11} defined by

$$\Delta\tau_1 = A_{11}\Delta\tau_0. \quad \text{(B.1)}$$

In the elastic regime, evolution of v is given by Eq. (26) which can be written as

$$v(\tau_0,\tau) = \mathrm{e}^{-\varsigma(\tau-\tau_0)}\left[A_1\cos\left(\sqrt{1-\varsigma^2}(\tau-\tau_0)\right) + B_1\sin\left(\sqrt{1-\varsigma^2}(\tau-\tau_0)\right)\right] + C_1\cos\omega\tau + D_1\sin\omega\tau$$
$$\text{with } A_1 = -1 - C_1\cos\omega\tau_0 - D_1\sin\omega\tau_0,\ B_1 = \frac{A_1\varsigma + C_1\omega\sin\omega\tau_0 - D_1\omega\cos\omega\tau_0}{\sqrt{1-\varsigma^2}},$$
$$C_1 = f_0\frac{1-\omega^2}{(1-\omega^2)^2+4\omega^2\varsigma^2},\ D_1 = f_0\frac{2\varsigma\omega}{(1-\omega^2)^2+4\omega^2\varsigma^2}. \quad \text{(B.2)}$$

A perturbation approach is used:

$$v(\tau_0,\tau_1) = 1,$$
$$v(\tau_0+\Delta\tau_0,\tau_1+\Delta\tau_1) = v(\tau_0,\tau_1) + \Delta\tau_0\frac{\partial v(\tau_0,\tau_1)}{\partial\tau_0} + \Delta\tau_1\frac{\partial v(\tau_0,\tau_1)}{\partial\tau_1} = 1. \quad \text{(B.3)}$$

The second equation of (B.3) leads to the determination of

$$A_{11} = -\frac{\partial v(\tau_0,\tau_1)/\partial\tau_0}{\partial v(\tau_0,\tau_1)/\partial\tau_1}. \quad \text{(B.4)}$$

A_{11} is finally calculated as

$$A_{11} = \frac{1+f_0\cos\omega\tau_0}{\dot{u}_1\sqrt{1-\varsigma^2}}\,\mathrm{e}^{-\varsigma(\tau_1-\tau_0)}\sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right), \quad \text{(B.5)}$$

where $\dot{u}_1$ is the displacement rate at the end of the elastic phase. $\dot{u}_1$ can be expressed by

$$\dot{u}_1 = \mathrm{e}^{-\varsigma(\tau_1-\tau_0)}\left[\left(-\varsigma A_1 + B_1\sqrt{1-\varsigma^2}\right)\cos\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\right.$$
$$\left.+\left(-\varsigma B_1 - A_1\sqrt{1-\varsigma^2}\right)\sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\right] - C_1\omega\sin\omega\tau_1 + D_1\omega\cos\omega\tau_1. \quad \text{(B.6)}$$

It is easy moreover to expand this term:

$$\Delta\dot{u}_1 = \frac{\partial\dot{u}_1(\tau_0,\tau_1)}{\partial\tau_0}\Delta\tau_0 + \frac{\partial\dot{u}_1(\tau_0,\tau_1)}{\partial\tau_1}\Delta\tau_1. \quad \text{(B.7)}$$

The term A_{31} can be deduced:

$$A_{31} = \frac{\partial\dot{u}_1(\tau_0,\tau_1)}{\partial\tau_0} + A_{11}\frac{\partial\dot{u}_1(\tau_0,\tau_1)}{\partial\tau_1}. \quad \text{(B.8)}$$

The terms of Eq. (B.8) are detailed below:

$$\frac{\partial \dot{u}_1(\tau_0,\tau_1)}{\partial \tau_1} = -1 - 2\varsigma \dot{u}_1 + f_0 \cos \omega\tau_1 \tag{B.9}$$

and

$$\frac{\partial \dot{u}_1(\tau_0,\tau_1)}{\partial \tau_0} = \left(1 + f_0 \cos \omega\tau_0\right) e^{-\varsigma(\tau_1-\tau_0)} \left[-\cos\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right) + \frac{\varsigma}{\sqrt{1-\varsigma^2}} \sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\right]. \tag{B.10}$$

The same reasoning can be applied to the determination of B_{11} and B_{13}, obtained from the function $\dot{u}_2(\dot{u}_1,\tau_1,\tau_2)$:

$$\dot{u}_2(\dot{u}_1,\tau_1,\tau_2) = B_2 e^{-2\varsigma(\tau_2-\tau_1)} + C_2 \cos \omega\tau_2 + D_2 \sin \omega\tau_2 - \frac{1}{2\varsigma}$$

$$\text{with } C_2 = f_0 \frac{2\varsigma}{\omega^2+4\varsigma^2},\ D_2 = f_0 \frac{\omega}{\omega^2+4\varsigma^2},\ B_2 = \dot{u}_1 + \frac{1}{2\varsigma} - C_2 \cos \omega\tau_1 - D_2 \sin \omega\tau_1. \tag{B.11}$$

The perturbation approach leads to

$$B_{11} = -\frac{\partial \dot{u}_2(\dot{u}_1,\tau_1,\tau_2)/\partial \tau_1}{\partial \dot{u}_2(\dot{u}_1,\tau_1,\tau_2)/\partial \tau_2} \quad \text{and} \quad B_{13} = -\frac{\partial \dot{u}_2(\dot{u}_1,\tau_1,\tau_2)/\partial \dot{u}_1}{\partial \dot{u}_2(\dot{u}_1,\tau_1,\tau_2)/\partial \tau_2}. \tag{B.12}$$

These terms are calculated as following:

$$B_{11} = \frac{e^{-2\varsigma(\tau_2-\tau_1)}}{1+f_0 \cos \omega\tau_0}\left(1 + 2\varsigma\dot{u}_1 - f_0 \cos \omega\tau_1\right) \quad \text{and} \quad B_{13} = \frac{e^{-2\varsigma(\tau_2-\tau_1)}}{1+f_0 \cos \omega\tau_0}. \tag{B.13}$$

Some simplification occurs by noticing that

$$B_{11} = -B_{13}\frac{\partial \dot{u}_1(\tau_0,\tau_1)}{\partial \tau_1}. \tag{B.14}$$

R is then simplified by

$$R = B_{13}\frac{\partial \dot{u}_1(\tau_0,\tau_1)}{\partial \tau_0} = e^{-\varsigma\left(\tau_0-\tau_1+(2\pi/\omega)\right)}\left[-\cos\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right) + \frac{\varsigma}{\sqrt{1-\varsigma^2}} \sin\left(\sqrt{1-\varsigma^2}(\tau_1-\tau_0)\right)\right]. \tag{B.15}$$

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